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THE MOTION OF A CLOSELY-FITTING SPHERE IN A FLUID-FILLED TUBE

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Abstract—Singular perturbation techniques are used to investigate the slow, asymmetric flow around a sphere positioned eccentrically within a long, circular, cylindrical tube filled with viscous fluid. The results apply to situations in which the sphere occupies virtually the entire cross section of the cylinder, so that the clearance between the particle and tube wall is everywhere small compared with both the sphere and tube radii. The technique is an improvement over conventional "lubrication-theory" analyses.

Asymptotic expansions, valid for small dimensionless clearances, are obtained for the hydrodynamic force, torque and pressure drop for flow past a stationary sphere, as well as for the case of a sphere translating or rotating in an otherwise quiescent fluid. These expansions are employed to predict the macroscopic behavior of both a neutrally-buoyant sphere suspended in a Poiseuille flow, and a sedimenting sphere in a vertical tube.

The results find application in capillary blood flow, pipeline transport of encapsulated materials, and falling-ball viscometers.

1. INTRODUCTION

The modeling of blood flow through individual capillaries has been the subject of numerous theoretical studies (Barnard, Lopez & Hellums 1968, Chen & Skalak 1970, Fitz-Gerald 1969, Hochmuth & Sutera 1970, Lighthill 1968, Wang & Skalak 1969). These studies all presume that blood plasma exhibits incompressible Newtonian properties, and that a lubricating layer of plasma surrounds the erythrocytes, thereby keeping the cells from direct contact with the vessel wall. If these premises are valid, the thickness of the plasma film between a red cell and the wall in a fine capillary would typically be small in comparison to the characteristic radial dimensions of either the capillary or the deformed cell. This view is supported by the *in vitro* measurements of Hochmuth, Marple & Sutera (1970). Consequently, previous investigators have invoked lubrication-theory arguments to render tractable the governing fluid–mechanical equations of motion (Barnard *et al.* 1968, Chen & Skalak 1970, Fitz-Gerald 1969, Hochmuth & Sutera 1970, Lighthill 1968). Solutions of the simplified equations which result resemble asymptotic expansions in terms of the clearance between the cell and capillary wall. Since these equations are simplified in a nonrigorous manner, lubrication-theory analyses cannot be relied upon to yield more than the correct leading term of a proper asymptotic expansion. In the present

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study, the asymptotic solution of the model flow equations is placed upon a firmer mathematical foundation. A systematic expansion procedure is developed, which draws upon unifying concepts from singular perturbation theory.

The model system consists of a rigid sphere and a Newtonian fluid within a long rigid cylindrical duct. The sphere diameter is comparable to the transverse dimension of the cylinder. Sphere and fluid densities may, or may not, be matched. Although the relationship to capillary blood flow (Brenner & Bungay 1971) furnished the original motivation for investigating this particular model, the results of the analysis have applicability to other fluid-particle systems—such as falling-ball viscometers, rotameters, and capsule transport in pipelines.

Utilization of singular perturbation theory to analyze particle-fluid behavior close to a rigid boundary was initially demonstrated by O'Neill & Stewartson (1967), Goldman, Cox & Brenner (1967), and Cox & Brenner (1967b). These authors treated the motion of a sphere in a fluid of semi-infinite extent, bounded by a plane wall. Their results were incorporated by Bungay & Brenner (1973) into an analysis of flow within a circular cylinder in which a relatively small sphere moves in close proximity parallel to the tube wall.

Singular perturbation analyses complement regular perturbation analyses of hydrodynamic interaction phenomena, e.g. the "method-of-reflections". Cox & Brenner (1967a) developed a general theory for the effect of rigid walls upon the hydrodynamic resistance of a translating-rotating particle. Their theory rests on regular expansions, which are valid in the limit where the particle is small compared with its distance from the boundaries girdling the flow. This same restriction applies to method-of-reflections results, such as those pertaining to a small sphere moving within a circular tube (Brenner & Happel 1958, Brenner 1966, 1970, Greenstein & Happel 1968).

When the fluid motion is axisymmetric, the range of validity of regular expansions can be extended to relatively large ratios of particle/wall diameters. Generation of such higher order terms by the method of reflections is typified by Bohlin's (1960) treatment of flow past a sphere whose center lies along the axis of a circular tube. Haberman & Sayre (1958), considering this same symmetrical configuration, arrived at more accurate expansions which were convergent for sphere/tube radius ratios between zero and 0·8. Wang & Skalak (1969) extended these single sphere results to an infinite train of identical, equally-spaced spheres, all of whose centers coincide with the tube axis. Convergence of the resulting expressions for drag and pressure drop was obtained up to radius ratios of 0·9, but the values near this upper limit were believed accurate only to two significant figures. Effect of particle spacing on hydrodynamic resistance was found to be weak in the train-of-spheres model of Wang & Skalak (1969). Each sphere behaved as an essentially isolated body for particle spacings exceeding approximately one tube diameter. The large sphere results, which are of interest in connection with the present investigation, indicate that particle–particle interactions are weak for all spacing distances. Chen & Skalak (1970) replaced the spheres with an infinite train of identical spheroids, whose symmetry axes lie along the tube axis. The flow remains axisymmetric in this case too, so that the influence of particle shape could be partially investigated to a similar order of accuracy. Particle interactions and wall effects for various related slow flow situations have been extensively reviewed

(Brenner 1966, Cox & Mason 1971, Goldsmith & Mason 1967, Happel & Brenner 1965).

Each of the above-cited theoretical investigations assumes that fluid inertial effects are small. A scheme for incorporating the additional effects stemming from fluid inertia has been developed by Cox & Brenner (1968), and discussed in detail by Brenner (1966), for spheres satisfying the method-of-reflections requirement of being small compared with their distance from the nearest wall. No comparable technique has yet been formulated for closely-fitting spheres. In Section 4 we suggest a scheme whereby the Reynolds number dependence can be incorporated into the singular perturbation analysis.

We begin with the creeping flow formulation of the problem of flow around a single, closely-fitting sphere in Section 2. The cylinder cross section is subsequently chosen to be that of a circular tube in Section 3. Singular perturbation expansions are derived in Section 4 for the particular case of a sphere undergoing a purely translational motion in an otherwise quiescent fluid. Corresponding results for a sphere in pure rotation, and for flow past a stationary sphere, are summarized in Sections 5 and 6. The situation most analogous to capillary blood flow is treated in Section 7, namely a neutrally-buoyant sphere suspended in a Poiseuille flow. We conclude with a discussion of the falling-ball viscometer in Section 8.

2. FORMULATION OF THE PROBLEM

Consider a long cylinder of constant, but arbitrary, cross section filled with a Newtonian fluid of viscosity μ and density ρ. The fluid moves in steady laminar flow at mean velocity V_m. Let a sphere of radius a now be suspended at some arbitrary lateral position in the cylinder, the mean velocity being maintained at the same value V_m as when the sphere is absent. In general, this requires the maintenance of a pressure difference between the two ends of the cylinder which exceeds that required in the absence of the sphere, i.e. the "Poiseuille" pressure drop for unidirectional flow through the cylinder. We seek to determine, *inter alia*, this additional pressure drop arising from the presence of the sphere in the otherwise rectilinear flow.

To formulate the problem explicitly, consider the portion of the cylinder (figure 1) bounded by the inside wall (S_w) and by the hypothetical planes at the duct inlet (S_i) and exit (S_e), situated at distances $l/2$ on either side of the sphere center. In the absence of the sphere, the unidirectional velocity and pressure fields are denoted by the field pair $(\mathbf{v}^o, p^o)$. Introduction of the sphere alters these to $(\mathbf{v}, p)$. Since the disturbance of the original flow produced by the sphere decays exponentially with distance up- and down-stream of the sphere, the length l may be chosen sufficiently large such that this disturbance has effectively been attenuated. Hence, on the inlet and exit planes, $\mathbf{v}(S_i) = \mathbf{v}(S_e) \equiv \mathbf{v}^o$. The local fluid pressure p is uniform across these planes, as it is for the undisturbed flow. Hence, we may unambiguously define the pressure drop, $\Delta p = p(S_i) - p(S_e)$. With $\Delta p^o = p^o(S_i) - p^o(S_e)$ the pressure drop in the absence of the sphere, the "additional" pressure drop due to the presence of the sphere in the flow may be defined as $\Delta p^+ = \Delta p - \Delta p^o$.

Relative to the fixed cylinder walls, the center of the sphere translates with a velocity $\mathbf{U}_o$. Simultaneously, the sphere rotates with angular velocity $\mathbf{\Omega}$. Consequently, the fluid in contact with the sphere surface (S_p) possesses a velocity

$$\mathbf{v} = \mathbf{U}_o + \mathbf{\Omega} \times \mathbf{r}_o \text{ on } S_p, \tag{2.1}$$

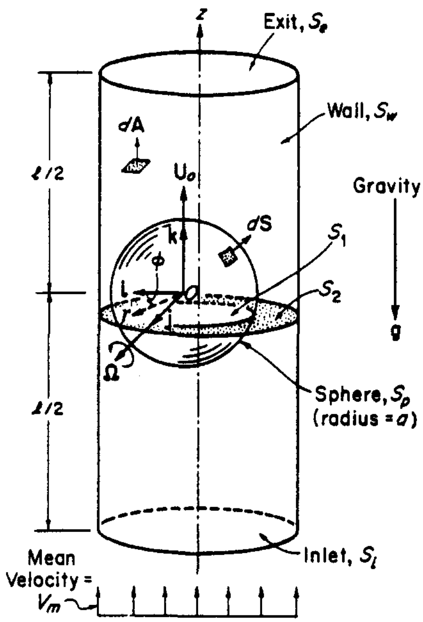

Figure 1. Closely-fitting sphere in Poiseuille flow through a vertical tube.

wherein $\mathbf{r}_o$ is a position vector originating at the sphere center. Similarly, at the cylinder wall,

$$\mathbf{v} = \mathbf{0} \text{ on } S_w. \qquad [2.2]$$

It is assumed at the outset that the relevant Reynolds number is sufficiently small compared with unity to permit neglect of the inertial terms in the Navier–Stokes equations, whence the basic dynamical and kinematical equations are taken to be

$$\mu \nabla^2 \mathbf{v} = \nabla p, \qquad [2.3]$$

$$\nabla \cdot \mathbf{v} = 0, \qquad [2.4]$$

with p the dynamic pressure. Later on, in Section 4, we will argue that since [2.3] need only be assumed to apply in the immediate proximity of the gap in order to calculate the important macroscopic parameters of the flow, the results to be obtained apply, in fact, to the complete Navier–Stokes equations. That is, the Reynolds number based upon gap width may be quite small, despite the fact that the Reynolds number based upon sphere size may be large (compared with unity).

In the absence of external forces, the sphere in creeping flow translates parallel to the cylinder walls, without experiencing lateral migration (Brenner & Bungay 1971). Thus, if $\mathbf{k}$ is a unit vector parallel to the generators of the cylinder,

$$\mathbf{U}_o = \mathbf{k} U_o.$$

Consider a cross-sectional plane perpendicular to $\mathbf{k}$ whose position, albeit arbitrary, is such that the plane intersects the sphere. As indicated in figure 1, this plane is composed of a portion S_1 lying within the sphere, and an annular portion S_2 occupying the gap between the sphere and the cylinder wall. If the cylinder cross-sectional area is denoted by A, and the areas of surfaces S_1 and S_2 by A_1 and A_2 respectively ($A = A_1 + A_2$), then the volumetric flow rate across S_2 relative to the containing vessel is

$$\int_{S_2} \mathbf{v} \cdot d\mathbf{A} = V_m A - U_o A_1. \qquad [2.5]$$

Here, $d\mathbf{A}$ is a directed differential element of surface area, and $V_m = A^{-1} \int_{S_1+S_2} \mathbf{v}^o \cdot d\mathbf{A}$ defines the mean velocity.

For purposes of the subsequent perturbation analysis, the linear system of governing equations [2.1]–[2.4] is best reformulated in terms of the "disturbance" fields,

$$\mathbf{v}^+ = \mathbf{v} - \mathbf{v}^o, \qquad [2.6]$$

and

$$p^+ = p - p^o. \qquad [2.7]$$

Since both $(\mathbf{v}, p)$ and $(\mathbf{v}^o, p^o)$ satisfy Stokes' equations, so must $(\mathbf{v}^+, p^+)$, whence

$$\mu \nabla^2 \mathbf{v}^+ = \nabla p^+, \qquad [2.8]$$

and

$$\nabla \cdot \mathbf{v}^+ = 0. \qquad [2.9]$$

The disturbance fields obey the boundary conditions,

$$\mathbf{v}^+ = \mathbf{U}_o + \mathbf{\Omega} \times \mathbf{r}_o - \mathbf{v}^o \text{ on } S_p, \qquad [2.10]$$

$$\mathbf{v}^+ = \mathbf{0} \text{ on } S_w, \qquad [2.11]$$

and

$$\mathbf{v}^+(S_i) = \mathbf{v}^+(S_e) \equiv \mathbf{0}. \qquad [2.12]$$

In addition, from [2.5], the velocity field is subject to the constraint,

$$\int_{S_2} \mathbf{v}^+ \cdot d\mathbf{A} = \int_{S_1} (\mathbf{v}^o - \mathbf{U}_o) \cdot d\mathbf{A}. \qquad [2.13]$$

To place [2.13] in a convenient form for applying the subsequent perturbation procedure, each of the two integration steps in the double integrals appearing therein must be effected separately. To accomplish this, define a circular cylindrical coordinate system, (r, ϕ, z), fixed relative to the vessel, with z axis passing through the sphere center, and lying parallel to the vessel wall. At the instant that the origin coincides with the sphere center o (figure 1), the radical position of the sphere surface is given by

$$r = r_p(z) \text{ on } S_p, \qquad [2.14]$$

in which

$$r_p = (a^2 - z^2)^{1/2}, \quad |z| \le a. \qquad [2.15]$$

The cylinder wall is to be represented by, $r = r_w[\phi]$. Since the cylinder cross section is uniform along its length, the radial position varies only with ϕ. The function $r_w[\phi]$ is left arbitrary for the present, except for the geometric restriction tkat $r_w[\phi] > a$ for all ϕ.

With (u^+, v^+, w^+) representing the appropriate circular cylindrical components of the velocity field $\mathbf{v}^+$, the continuity equation [2.9] becomes,

$$\frac{1}{r}\frac{\partial}{\partial r}(ru^+) + \frac{1}{r}\frac{\partial v^+}{\partial \phi} + \frac{\partial w^+}{\partial z} = 0.$$

Multiplying by $r\,\mathrm{d}r$ and integrating across the annular gap between the sphere and the wall gives, with the aid of boundary conditions [2.10] and [2.11],

$$\int_{r_p}^{r_w} \frac{\partial w^+}{\partial z} r\,\mathrm{d}r + \int_{r_p}^{r_w} \frac{\partial v^+}{\partial \phi}\mathrm{d}r - \Omega r_p z \cos\phi = 0,$$

where the scalar Ω is defined in [3.3]. It is understood that the restriction $|z| \leq a$ applies to this relation, as well as to subsequent equations derived from it. By employing Leibnitz' rule, the boundary conditions, and [2.15], the last equation can be rewritten as

$$\int_{r_p}^{r_w} w^+ r\,\mathrm{d}r + \frac{\partial}{\partial \phi}\int\int_{r_p}^{r_w} v^+\,\mathrm{d}r\,\mathrm{d}z = \int_0^{r_p} w^o r\,\mathrm{d}r + \frac{1}{2}U_o z^2 + q[\phi], \qquad [2.16]$$

in which $\mathbf{v}^o = \mathbf{k}w^o[r, \phi]$, and $q[\phi]$ is a function which arises in consequence of the indefinite integration with respect to z. Integrating each term of [2.16] from $\phi = -\pi$ to π gives

$$\int_{-\pi}^{\pi}\int_{r_p}^{r_w} w^+ r\,\mathrm{d}r\,\mathrm{d}\phi = \int_{-\pi}^{\pi}\int_0^{r_p} w^o r\,\mathrm{d}r\,\mathrm{d}\phi + U_o \pi z^2 + \int_{-\pi}^{\pi} q\,\mathrm{d}\phi.$$

This equation has been simplified by noting that, $v^+[\phi = -\pi] = v^+[\phi = +\pi]$. In vector form this becomes,

$$\int_{-\pi}^{\pi} q\,\mathrm{d}\phi = \int_{S_2} \mathbf{v}^+ \cdot \mathrm{d}\mathbf{A} - \int_{S_1} \mathbf{v}^o \cdot \mathrm{d}\mathbf{A} - U_o \pi z^2.$$

Substitution of [2.13] into the last equation yields,

$$\int_{-\pi}^{\pi} q\,\mathrm{d}\phi = -U_o \pi a^2. \qquad [2.17]$$

Equations [2.8]–[2.12], [2.16] and [2.17] constitute the system of equations describing the flow past a sphere within a cylinder. The latter two equations are continuity relationships, which replace the volumetric flow rate condition [2.13].

3. CYLINDER OF CIRCULAR CROSS-SECTION

We now specialize the general formulation of the previous section to the case where the cylinder is a circular tube of radius R_o. Let b denote the perpendicular distance from the sphere center to the tube axis. Clearly, $0 \leq b \leq R_o - a$, the lower bound corresponding to the concentric position, and the upper bound to the fully eccentric position, where the sphere contacts the wall. The radial position of the tube wall is readily found to be expressed by the relation,

$$r_w[\phi] = (R_o^2 - b^2 \sin^2\phi)^{1/2} - b\cos\phi. \qquad [3.1]$$

In the absence of the sphere the flow is Poiseuillian, whence the undisturbed velocity profile is given in the cylindrical coordinate system as

$$\mathbf{v}^o = 2\mathbf{k}V_m[1 - (b/R_o)^2 - 2(br/R_o^2)\cos\phi - (r/R_o)^2]. \qquad [3.2]$$

With the sphere present, there exist two mutually perpendicular planes of reflection symmetry, these being the cross-sectional plane ($z = 0$) through the sphere center, and the meridian plane ($\phi = 0, \pi, -\pi$) passing through both the sphere center and tube axis. It will be convenient to employ unit vectors **i** and **j**, which together with **k** form a right-handed, orthonormal set. The vector **i** is chosen to lie along the line of intersection of the two symmetry planes. A neutrally buoyant sphere occupying an eccentric location ($b \neq 0$) rotates. By virtue of the prevailing geometrical symmetry, the angular velocity vector characterizing this rotation is necessarily of the form

$$\mathbf{\Omega} = \mathbf{j}\Omega, \qquad [3.3]$$

which lies parallel to the local vorticity vector of the undisturbed flow. Equation [3.3] applies to motion induced by external forces acting on the sphere in a direction parallel to the tube axis; e.g. a non-neutrally buoyant (but homogeneous) sphere in a vertical tube subject to gravity.

For creeping flow with external forces directed parallel to the tube axis, prevailing symmetry conditions insure that the sphere experiences only a hydrodynamic force **F** and hydrodynamic torque $\mathbf{T}_o$, such that

$$\mathbf{F} = \mathbf{k}F, \qquad [3.4]$$

$$\mathbf{T}_o = \mathbf{j}T_o. \qquad [3.5]$$

The subscript o indicates that the torque is to be evaluated about the sphere center. For a proof of [3.4], see Cox & Mason (1971). A proof of [3.5] may be constructed along similar lines.

Linearity of Stokes' equations and of the boundary conditions implies that the hydrodynamic force, torque, and additional pressure drop force are linear functions of the three characteristic velocities. This linearity can be concisely expressed by the matrix equation

$$\begin{pmatrix} F \\ T_o/a \\ \Delta p^+ A \end{pmatrix} = -\mu a \begin{pmatrix} K_t & K_r & K_s \\ L_t & L_r & L_s \\ M_t & M_r & M_s \end{pmatrix} \begin{pmatrix} U_o \\ \Omega a \\ -V_m \end{pmatrix}. \qquad [3.6]$$

The intrinsic scalar resistance coefficients, K, L and M, appearing in the 3×3 matrix are dimensionless functions, dependent only on the sphere-tube geometry, i.e. upon a/R_o and b/R_o. In particular, they are independent of the viscosity, of the velocities U_o, Ω and V_m, and of the absolute sizes of the sphere and tube. These coefficients are all positive, with the possible exception of K_r and L_t, which may assume either sign. The latter two elements are equal:

$$K_r = L_t. \qquad [3.7]$$

Though this equality was originally derived by Brenner (1964) for particle motion in an unbounded medium, it holds also for bounded flows. Bungay & Brenner (1973) have shown that the resistance coefficient matrix in [3.6] is both positive definite and symmetric. This symmetry is manifested by the additional identities,

$$M_t = K_s, \quad M_r = L_s. \qquad [3.8]$$

4. SPHERE TRANSLATING THROUGH A QUIESCENT FLUID

Detailed derivations of the asymptotic values of the resistance coefficients in [3.6] for small gap widths are given by Bungay (1970). This singular perturbation expansion procedure will be illustrated by examining the purely translational motion of the sphere ($\Omega = 0$) through an otherwise quiescent medium ($V_m = 0$). Such motion can be achieved physically in a vertical tube by the settling of an inhomogeneous sphere, which was loaded to prevent rotation.

Outer region equations

Define the following dimensionless "outer" variables, denoted by a prime:

$$\mathbf{v}^+ = U_o \mathbf{v}', \qquad [4.1]$$

$$p^+ = (\mu U_o/a)p', \quad \mathbf{r}_o = (ar', \phi, az'). \qquad [4.2]$$

In terms of these, the governing equations for the outer region, away from the gap, as obtained from [2.8]–[2.12], [2.16] and [2.17], are

$$\nabla'^2 \mathbf{v}' - \nabla' p' = 0, \qquad [4.3]$$

$$\nabla' \cdot \mathbf{v}' = 0, \qquad [4.4]$$

$$\mathbf{v}' = \mathbf{k} \text{ on } S_p, \qquad [4.5]$$

$$\mathbf{v}' = \mathbf{0} \text{ on } S_w, \qquad [4.6]$$

$$\mathbf{v}' \to \mathbf{0} \text{ as } |z'| \to \infty, \qquad [4.7]$$

$$\int_{r'_p}^{r'_w} w' r' \, dr' + \frac{\partial}{\partial \phi} \int \int_{r'_p}^{r'_w} v' \, dr' \, dz' = \frac{1}{2}(z')^2 + q'[\phi], \quad |z'| \le 1, \qquad [4.8]$$

and

$$\int_{-\pi}^{\pi} q' \, d\phi = -\pi, \qquad [4.9]$$

in which

$$\nabla' = a\nabla, \quad q' = q/U_o a^2.$$

Inner region variables

The equations describing the fluid motion in the inner region, in the proximity of the small gap, are obtained by introducing two dimensionless geometric parameters, representing respectively the lateral position of the sphere,

$$e = b/(R_o - a), \quad 0 \le e \le 1, \qquad [4.10]$$

and the clearance between the sphere and wall,

$$\varepsilon = (R_o - a)/a, \quad 0 < \varepsilon < \infty. \qquad [4.11]$$

Closely-fitting spheres are characterized by $\varepsilon \ll 1$. Consequently, ε serves as a small perturbation parameter.

A new radial variable may be defined for the gap region between the sphere and the wall as

$$x' = 1 + \varepsilon(1 - e\cos\phi) - r'. \qquad [4.12]$$

In terms of this new variable, the equation of the sphere surface adopts the form,

$$x'_p = 1 + \varepsilon(1 - e\cos\phi) - [1 - (z')^2]^{1/2}, \quad |z'| \leq 1.$$

For $|z'| \ll 1$ the square root term in the above equation may be replaced by its power series representation, yielding

$$x'_p = \varepsilon(1 - e\cos\phi) + \tfrac{1}{2}(z')^2 + \tfrac{1}{8}(z')^4 + \tfrac{1}{16}(z')^6 + \cdots.$$

This expansion suggests "stretching" the coordinates by defining the new coordinate variables,

$$X = x'/\varepsilon, \qquad [4.13]$$

and

$$Z = z'/\varepsilon^{1/2}. \qquad [4.14]$$

No rescaling of the angular variable is necessary, owing to the fact that the primary variations in gap width occur in the x' and z' directions. Accordingly, we may put $\Phi = \phi$. These yield

$$X_p = H + \tfrac{1}{8}\varepsilon Z^4 + \tfrac{1}{16}\varepsilon^2 Z^6 + \cdots \qquad [4.15]$$

as the equation describing the sphere surface. Here,

$$H[\Phi, Z] = \tau^2 + \tfrac{1}{2}Z^2, \qquad [4.16]$$

and

$$\tau[\Phi] = (1 - e\cos\Phi)^{1/2}, \qquad [4.17]$$

are frequently recurring functions.

The expansion of the wall surface [3.1] becomes,

$$X_w = \tfrac{1}{2}\varepsilon(e\sin\Phi)^2 - \tfrac{1}{2}\varepsilon^2(e\sin\Phi)^2 + \cdots. \qquad [4.18]$$

Comparison of [4.15] with [4.18] reveals that H constitutes the dimensionless separation between the two surfaces to terms of zero order in ε, when expressed in inner coordinates.

Introduction of [4.12]–[4.14] into [4.8] yields

$$\varepsilon \int_{X_w}^{X_p} w' \, dX = q' + O(\varepsilon),$$

with q' taken to be of zero order in ε. Since X_p is also of zero order, the integral in this equation can be made of similar order by defining a stretched axial velocity component as

$$W = \varepsilon w'. \qquad [4.19]$$

In the continuity equation [4.4] the conversion of outer variables to those inner variables already defined suggests the additional definitions,

$$U = \varepsilon^{1/2}u', \qquad [4.20]$$

$$V = \varepsilon^{1/2}v'. \qquad [4.21]$$

Lastly, the **k**-component of [4.3], when transformed to inner variables, leads to the following rescaling of the pressure field:

$$P = \varepsilon^{5/2}p'. \qquad [4.22]$$

Inner expansions

Through the stretching procedure, inner velocity and pressure fields have been defined which are each of zero order in ε in the limit as $\varepsilon \to 0$. They still depend, however, upon ε in an unknown manner. It is now assumed, subject to *a posteriori* justification, that for $\varepsilon \ll 1$ these fields possess the following power series expansions:

$$U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \cdots, \qquad [4.23]$$

$$V = V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + \cdots, \qquad [4.24]$$

$$W = W_0 + \varepsilon W_1 + \varepsilon^2 W_2 + \cdots, \qquad [4.25]$$

and

$$P = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \cdots, \qquad [4.26]$$

in which the subscripted fields are independent of ε, depending only upon (X, Φ, Z) and e. The quantity q' varies with ε. Its asymptotic expansion is similarly assumed to be

$$q' = Q_0 + \varepsilon Q_1 + \varepsilon^2 Q_2 + \cdots,$$

wherein each of the ordered perturbations, Q_k, are functions of Φ.

The outer variables in the differential equations [4.3] and [4.4] may each be converted to inner variables via the definitions [4.12]–[4.22]. Following substitution of the asymptotic expansions [4.23]–[4.26] into the resulting equations, terms of like order in ε may be collected, and the separate terms of $O(\varepsilon^n)$ thereby obtained each set equal to zero for $n = 0, 1, 2, \ldots$. This process produces the following ordered set of differential equations. For the zero order terms,

$$\partial P_0/\partial X = 0, \qquad [4.27]$$

$$\partial^2 V_0/\partial X^2 - \partial P_0/\partial \Phi = 0, \qquad [4.28]$$

$$\partial^2 W_0/\partial X^2 - \partial P_0/\partial Z = 0, \qquad [4.29]$$

$$\partial U_0/\partial X - \partial W_0/\partial Z = 0. \qquad [4.30]$$

Similarly, the first order terms yield

$$\partial P_1/\partial Z = -\partial^2 U_0/\partial X^2, \qquad [4.31]$$

$$\partial^2 V_1/\partial X^2 - \partial P_1/\partial \Phi = \partial V_0/\partial X + (X - \tau^2)\partial^2 V_0/\partial X^2 - \partial^2 V_0/\partial Z^2 - e \sin \Phi\, \partial^2 U_0/\partial X^2, \qquad [4.32]$$

$$\partial^2 W_1/\partial X^2 - \partial P_1/\partial Z = \partial W_0/\partial X - \partial^2 W_0/\partial Z^2, \tag{4.33}$$

$$\partial U_1/\partial X - \partial W_1/\partial Z = U_0 + e\sin\Phi\,\partial V_0/\partial X + \partial V_0/\partial\Phi. \tag{4.34}$$

This procedure may be applied up to any desired order in ε.

This same process may be repeated for the inner-region boundary conditions [4.5], [4.6], [4.8] and [4.9]. However, such a procedure fails to be complete since the boundary conditions would still be implicit functions of ε in consequence of the surface equations [4.15] and [4.18]. To render the ε-dependence from the wall surface explicit, velocity boundary conditions can be transferred to $X = 0$.

Transfer of boundary conditions

Let $\Psi[X]$ be one of the three stretched velocity components. The Taylor series expansion of each perturbation field Ψ_k about $X = 0$, when evaluated at $X = X_w$, leads to

$$\Psi[X_w] = \Psi_0[0] + \varepsilon(\Psi_1[0] + \tfrac{1}{2}e^2 \sin^2\Phi\, \mathrm{d}\Psi_0[0]/\mathrm{d}X) + O(\varepsilon^2).$$

Similarly, the boundary conditions on the sphere surface can be transferred to $X = H$ by use of the expansion,

$$\Psi[X_p] = \Psi_0[H] + \varepsilon(\Psi_1[H] + \tfrac{1}{2}[H - \tau^2]^2\, \mathrm{d}\Psi_0[H]/\mathrm{d}X) + O(\varepsilon^2),$$

wherein the definition of H given in [4.16] has been used to replace Z in the coefficients. The concept of transfer of boundary conditions in perturbation procedures is discussed by Van Dyke (1964).

The integration limits in the continuity condition [4.8] must be brought into agreement with the new hypothetical boundaries, $X = 0$ and $X = H$. This is effected with the aid of the following series representation, obtained by Taylor expansion and application of Leibnitz' rule:

$$\int_{X_w}^{X_p} \Psi_k[X]\,\mathrm{d}X = \int_0^H \Psi_k[X]\,\mathrm{d}X + \varepsilon(\tfrac{1}{2}[H - \tau^2]\Psi_k[H] - \tfrac{1}{2}e^2\sin^2\Phi\,\Psi_k[0]) + O(\varepsilon^2).$$

Setting the coefficient of the zero order term in each of the expanded, inner boundary conditions equal to zero gives,

$$U_0 = V_0 = W_0 = 0 \text{ at } X = 0, \tag{4.35}$$

$$U_0 = V_0 = W_0 = 0 \text{ at } X = H, \tag{4.36}$$

$$\int_0^H W_0\,\mathrm{d}X - Q_0 = 0, \tag{4.37}$$

$$\int_{-\pi}^{\pi} Q_0\,\mathrm{d}\Phi = -\pi. \tag{4.38}$$

Likewise, the set of first order boundary conditions thereby obtained is,

$$\left.\begin{aligned} U_1 &= -\tfrac{1}{2}e^2 \sin^2\Phi\, \partial U_0/\partial X, \\ V_1 &= -\tfrac{1}{2}e^2 \sin^2\Phi\, \partial V_0/\partial X, \\ W_1 &= -\tfrac{1}{2}e^2 \sin^2\Phi\, \partial W_0/\partial X \end{aligned}\right\} \quad \text{at } X = 0, \qquad [4.39]$$

$$\left.\begin{aligned} U_1 &= -\tfrac{1}{2}(H - \tau^2)^2\, \partial U_0/\partial X, \\ V_1 &= -\tfrac{1}{2}(H - \tau^2)^2\, \partial V_0/\partial X, \\ W_1 &= 1 - \tfrac{1}{2}(H - \tau^2)^2\, \partial W_0/\partial X \end{aligned}\right\} \quad \text{at } X = H, \qquad [4.40]$$

$$\int_0^H W_1\, \mathrm{d}X - Q_1 = H - \tau^2 - \frac{\partial}{\partial\Phi}\int\int_0^H V_0\, \mathrm{d}X\, \mathrm{d}Z + (H - \tau^2)\int_0^H W_0\, \mathrm{d}X - \\ - \int_0^H \int_0^X W_0[\theta, \Phi, Z]\, \mathrm{d}\theta\, \mathrm{d}X, \qquad [4.41]$$

$$\int_{-\pi}^{\pi} Q_1\, \mathrm{d}\Phi = 0. \qquad [4.42]$$

Inner perturbation fields

The ordered systems of perturbation equations must be solved sequentially. Equations [4.27]–[4.30] and [4.35]–[4.38], representing the zero order set, are analogous to Reynolds lubrication-theory equations. Each successive higher-order system represents an additional improvement over the accuracy of the conventional lubrication theory approximation.

Each set of perturbation differential equations has been written in a form whereby only the unknown quantities sought appear on the left-hand side of each equation. Comparison of the structure of the first-order system with that of the zero-order system reveals that the left-hand sides of the corresponding pairs of equations are identical in form. The right-hand sides contain only constants or functions known from the solutions of the lower-order systems. Thus, the procedure for solving the zero-order system applies generally to the solution of every other ordered set of equations—but the extent of the algebraical manipulations increases significantly as one progresses to higher-order approximations.

Though the ordered systems of equations generally require the simultaneous solution of four coupled partial differential equations, their structure is such that they can be treated as easily as though the equations involved only ordinary derivatives. The solution procedure will be demonstrated in detail for the zero-order system.

Zero-order inner solution

Equation [4.27] indicates that the dynamic pressure does not vary across the gap. This is the usual lubrication-theory assumption, here rigorously substantiated by the perturbation analysis. A partial integration of [4.27] gives for the pressure distribution,

$$P_0[X, \Phi, Z] = f_0[\Phi, Z]. \qquad [4.43]$$

Substitution of the above into [4.28] and [4.29] then gives

$$\partial^2 V_0/\partial X^2 = \partial f_0/\partial\Phi, \quad \partial^2 W_0/\partial X^2 = \partial f_0/\partial Z.$$

Integration of the latter equations with respect to X, and utilization of boundary conditions [4.35] and [4.36], yields

$$V_0 = \tfrac{1}{2}(X^2 - HX)\partial f_0/\partial\Phi, \qquad [4.44]$$

$$W_0 = \tfrac{1}{2}(X^2 - HX)\partial f_0/\partial Z. \qquad [4.45]$$

The continuity equation [4.30] then provides the radial velocity component,

$$U_0 = \frac{\partial}{\partial Z}\left[\left(\frac{1}{6}X^3 - \frac{1}{4}HX^2\right)\frac{\partial f_0}{\partial Z}\right]. \qquad [4.46]$$

On substituting for W_0 in [4.37], one obtains

$$\partial f_0/\partial Z = -12Q_0/H^3. \qquad [4.47]$$

Integration from $Z = -\infty$ to $+\infty$, with use of the definition of H given in [4.16], then yields

$$f_0[\Phi, -\infty] - f_0[\Phi, +\infty] = 9\pi\sqrt{2}Q_0/2\tau^5.$$

Hence, from [4.43] it follows that

$$P_0[X, \Phi, -\infty] - P_0[X, \Phi, +\infty] = 9\pi\sqrt{2}Q_0/2\tau^5.$$

The left-hand side, ΔP_0, say, representing the zero-order contribution to the additional pressure drop arising from the presence of the sphere, is necessarily constant, independent of X and Φ. Hence,

$$Q_0 = 2\tau^5\Delta P_0/9\pi\sqrt{2}. \qquad [4.48]$$

When substituting into the remaining boundary condition [4.38], this yields

$$\Delta P_0 = -\tfrac{9}{4}\pi\sqrt{2}\eta_0, \qquad [4.49]$$

in which η_0 is a function of the lateral position of the sphere in the tube, given by

$$\eta_0[e] = 2\pi\left[\int_{-\pi}^{\pi}\tau^5\,d\Phi\right]^{-1} \qquad [4.50]$$

The integral appearing in [4.50] was originally evaluated by Christopherson & Dowson (1959) via a method which is outlined in the accompanying Appendix. Variation of the function with the lateral position of the sphere is shown in figure 2. Integration of [4.47] yields

$$f_0 = \frac{9}{4}\sqrt{2}\eta_0\left[\arctan\frac{Z}{\tau\sqrt{2}} + \frac{Z\tau^3}{H\sqrt{2}} + \frac{2Z\tau^4}{3H^2\sqrt{2}}\right], \qquad [4.51]$$

from which all of the zero-order inner fields can now be obtained.

First-order inner fields

Expressions for the first-order inner velocity and pressure fields were similarly derived (Bungay 1970) for a non-rotating sphere translating at an eccentric lateral position. For

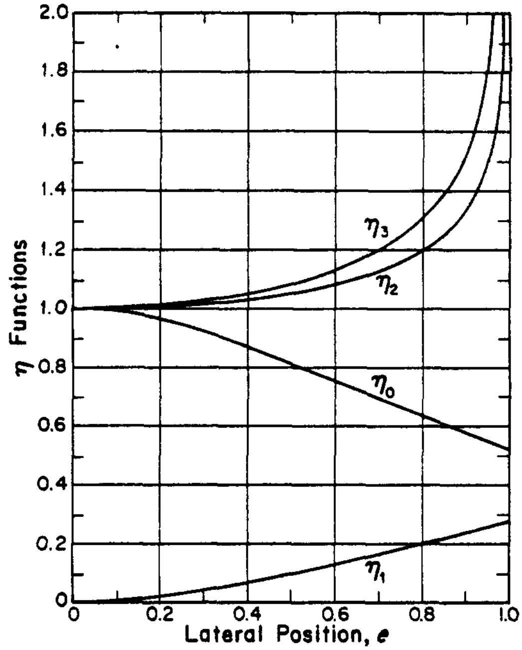

Figure 2. η Functions.

present purposes it suffices to present only the result obtained for the first-order pressure drop,

$$\Delta P_1 = -\tfrac{9}{4}\pi\sqrt{2}\eta_0[\tfrac{37}{60} + \eta_1]. \qquad [4.52]$$

The eccentricity function η_1 possesses a zero value for a concentric sphere; i.e. $\eta_1[0] = 0$. For all lateral positions, the value of η_1 obtained from the relation

$$\eta_1[e] = \frac{193}{60} - \frac{\eta_0}{120\pi}\left[90(1 - e^2)\int_{-\pi}^{\pi} \tau^3\, d\Phi + 103\int_{-\pi}^{\pi} \tau^7\, d\Phi\right]. \qquad [4.53]$$

An expression for η_1 in terms of tabulated functions is derived in the Appendix.

Second-order inner fields

Due to algebraic complexities, the second-order fields were calculated only for the axisymmetric case, where the sphere translates with its center along the tube axis. The second-order pressure drop obtained for this case is

$$\Delta P_2[e = 0] = \frac{9}{4}\pi\sqrt{2}\left(\frac{11{,}467}{50{,}400}\right) \equiv 2.2744. \qquad [4.54]$$

Resistance coefficients

From the defining equation [3.6] for the pressure drop resistance coefficient, along with definitions cited in [4.2] and [4.22], one can construct the asymptotic expansion,

$$M_t = -\pi(1 + \varepsilon)^2\varepsilon^{-5/2}(\Delta P_0 + \varepsilon\Delta P_1 + \varepsilon^2\Delta P_2) + O(1). \qquad [4.55]$$

In conjunction with [4.49], [4.52] and [4.54] this leads to the relation

$$M_t = \frac{9}{4}\pi^2\sqrt{2}\eta_0\varepsilon^{-5/2}\left[1 + \varepsilon\left(\frac{157}{60} + \eta_1\right) + \varepsilon^2\left(\frac{101{,}093}{50{,}400} + \eta_t\right)\right] + m_t + O(\varepsilon^{1/2}), \qquad [4.56]$$

in which the, as yet, undetermined dependence of the terms of $O(\varepsilon^{1/2})$ and $O(1)$ upon the lateral position of the sphere is represented by the functions $\eta_t = \eta_t[e]$ and $m_t = m_t[e]$, respectively. Although the functions η_t and m_t are unknown, their values should be finite for all e. In particular, η_t vanishes for the axisymmetric case, $e = 0$. The function m_t, which is of $O(1)$ with respect to ε, cannot be evaluated without detailed knowledge of the solution of the outer region equations.

The hydrodynamic force exerted by the fluid on the sphere is $\mathbf{F} = \int_{S_p} d\mathbf{S}\cdot\boldsymbol{\pi}$, in which $\boldsymbol{\pi} = -\mathbf{I}p + \mu[\nabla\mathbf{v} + (\nabla\mathbf{v})\dagger]$ is the Newtonian pressure tensor. The undisturbed flow contribution $\boldsymbol{\pi}^o$, appearing in the decomposition $\boldsymbol{\pi} = \boldsymbol{\pi}^o + \boldsymbol{\pi}^+$, is a regular field within the fluid volume bounded externally by the surface S_p, and hence makes no contribution to the force integral. Expressed in cylindrical coordinates, the scalar force is thus given in terms of the disturbance fields by the relation

$$F = 2\int_{-\pi}^{\pi}\int_0^a\left[\mu r\left(\frac{\partial u^+}{\partial z} + \frac{\partial w^+}{\partial r}\right) + z\left(-p^+ + 2\mu\frac{\partial w^+}{\partial z}\right)\right]_{r=r_p} dz\, d\phi. \qquad [4.57]$$

The fact that the integrand is an even function of z has been used to halve the interval of integration.

The domain of the z integration in [4.57] may be further divided into two portions, corresponding to the two regions of the perturbation solution. Let $z/a = \zeta \ll 1$ be an arbitrary plane lying in the common region of validity of both the inner and outer expansions. To obtain the outer contribution to the resistance coefficient, [4.57] is first rewritten in the dimensionless variables appropriate to that region. The defining relation [3.6] between the force and the resistance coefficient is then employed to produce the expression

$$K_t^{(o)} = -2\int_{-\pi}^{\pi}\int_{\zeta}^{1}\left[r'\left(\frac{\partial u'}{\partial z'} + \frac{\partial w'}{\partial r'}\right) + z'\left(-p' + 2\frac{\partial w'}{\partial z'}\right)\right]_{r'=r'_p} dz'\, d\phi. \qquad [4.58]$$

Outer contributions to the force that are singular in ε arise only from the pressure field, since the outer velocity components are each of zero order in ε. Equation [4.58] thus becomes,

$$K_t^{(o)} = -\left(\frac{\Delta P_0}{\varepsilon^{5/2}} + \frac{\Delta P_1}{\varepsilon^{3/2}} + \frac{\Delta P_2}{\varepsilon^{1/2}}\right)\int_{-\pi}^{\pi}\int_{\zeta}^{1} z'\, dz'\, d\phi + O(1). \qquad [4.59]$$

Physically, this result can be interpreted as indicating that the dominant contribution to the force is not the shear stresses *per se*, but rather the difference between upstream and downstream pressures acting on the sphere. Hence, from previous pressure drop results one finds that

$$K_t^{(o)} = \frac{9\pi^2\sqrt{2}\eta_0}{4\varepsilon^{5/2}}\left[1 + \varepsilon\left(\frac{37}{60} + \eta_1\right) + \varepsilon^2\left(-\frac{11{,}467}{50{,}400} - 2\eta_1 + \eta_t\right)\right](1 - \zeta^2) + O(1).$$

The contribution from the inner region is similarly to be obtained from the relation

$$K_t^{(i)} = -2\varepsilon^{-3/2}\int_{-\pi}^{\pi}\int_{0}^{\zeta/\varepsilon^{1/2}}\left[(1-\varepsilon Z^2)^{1/2}\left(\varepsilon\frac{\partial U}{\partial Z}-\frac{\partial W}{\partial X}\right)+Z\left(-P+2\varepsilon\frac{\partial W}{\partial Z}\right)\right]_{X=X_p} dZ\, d\Phi.$$

Use of asymptotic expansions [4.23]–[4.26], and transference of the integrand evaluation to $X = H$, ultimately leads to

$$K_t^{(i)} = 2\varepsilon^{-3/2}\left[\int_{-\pi}^{\pi}\int_{0}^{\zeta/\varepsilon^{1/2}}\left(ZP_0+\frac{\partial W_0}{\partial X}\right)_{X=H} dZ\, d\Phi + \varepsilon\int_{-\pi}^{\pi}\int_{0}^{\zeta/\varepsilon^{1/2}}\left(ZP_1+\frac{\partial W_1}{\partial X}\right.\right.$$
$$\left.\left. -\frac{\partial U_0}{\partial Z}+[1-H]\frac{\partial W_0}{\partial X}-2Z\frac{\partial W_0}{\partial Z}\right)_{X=H} dZ\, d\Phi + O(\varepsilon^2)\right]. \quad [4.60]$$

Utilization of only the zero-order fields yields

$$K_t^{(i)} = \sqrt{2}\,\eta_0\varepsilon^{-3/2}\int_{-\pi}^{\pi}\left\{\tau^2\left[\left(\frac{9}{2}\xi+\frac{3}{2}\right)\arctan\xi+\frac{9}{2}\xi\right]+O(\varepsilon)\right\}_{\xi=\zeta/\tau(2\varepsilon)^{1/2}} d\Phi.$$

Expand† the integrand as a power series of $1/\xi$, and perform the integration to obtain

$$K_t^{(i)} = \tfrac{9}{4}\pi^2\sqrt{2}\,\eta_0\varepsilon^{-5/2}[\zeta + O(\varepsilon)].$$

Inclusion of higher-order fields yields the more accurate result,

$$K_t^{(i)} = \frac{9}{4}\pi^2\sqrt{2}\,\eta_0\varepsilon^{-5/2}\left(\zeta^2+\varepsilon\left[\frac{2}{3}+\left(\frac{37}{60}+\eta_1\right)\zeta^2\right]\right.$$
$$\left.+\varepsilon^2\left[\frac{16\eta_2}{9\eta_0}+\frac{1}{30}+\frac{2}{3}\eta_1-\frac{7}{15}e^2+\left(-\frac{11{,}467}{50{,}400}-2\eta_1+\eta_t\right)\zeta^2\right]\right)+O(1), \quad [4.61]$$

in which

$$\eta_2[e] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\Phi}{(1-e\cos\Phi)^{1/2}}. \quad [4.62]$$

This integral is evaluated in the Appendix.

The pertubation expansion of the resistance coefficient is the sum of the inner and outer contributions,

$$K[\varepsilon] = K^{(o)}[\varepsilon;\zeta] + K^{(i)}[\varepsilon;\zeta]. \quad [4.63]$$

This sum must not contain ζ, since this quantity is arbitrarily defined. Adding together the

† Following the reasoning of Goldman, Cox & Brenner (1967), ζ may be set equal to $B\varepsilon^{\alpha}$, where B is any positive constant, and α is a constant such that $0 < \alpha < \frac{1}{2}$. Thus, the outer variable ζ must tend to zero as $\varepsilon \to 0$. On the other hand, the inner variable $z = \zeta/\varepsilon^{1/2} = B\varepsilon^{\alpha-1/2}$ must tend to infinity in the same limit. The asymptotic agreement between a typical inner solution $\Psi[Z]$, and the complementary outer solution, $\psi[z']$, in the region of overlap can be expressed by the asymptotic formula

$$\Psi[Z\to\infty] \sim \psi[z'\to 0].$$

The foregoing is a statement of the so-called "matching principle" (Van Dyke 1964). Here, we need only note that $\xi = (B/\tau\sqrt{2})\varepsilon^{\alpha-1/2} \to \infty$ as $\varepsilon \to 0$.

calculated inner and outer terms gives

$$K_t=\frac{9}{4}\pi^2\sqrt{2}\,\eta_0\varepsilon^{-5/2}\left[1+\varepsilon\left(\frac{77}{60}+\eta_1\right)+\varepsilon^2\left(-\frac{9787}{50{,}400}-\frac{7}{15}e^2-\frac{4}{3}\eta_1+\frac{16\eta_2}{9\eta_0}+\eta_t\right)\right]+k_t+O(\varepsilon^{1/2}), \qquad [4.64]$$

whence it is seen that ζ properly cancels.

The hydrodynamic torque acting on the sphere about its center,

$$\mathbf{T}_o = \int_{S_p} \mathbf{r}_o \times (d\mathbf{S}\cdot\boldsymbol{\pi}^+),$$

when formulated in cylindrical coordinates is given in scalar form by the expression

$$T_o = 2\mu\int_{-\pi}^{\pi}\int_0^a\left\{\left[2rz\left(\frac{\partial u^+}{\partial r}-\frac{\partial w^+}{\partial z}\right)-(r^2-z^2)\left(\frac{\partial u^+}{\partial z}+\frac{\partial w^+}{\partial r}\right)\cos\phi\right.\right.$$
$$\left.\left.-\left[rz\left(r\frac{\partial}{\partial r}\left\{\frac{v^+}{r}\right\}+\frac{1}{r}\frac{\partial u^+}{\partial\phi}\right)+z^2\left(\frac{\partial v^+}{\partial z}+\frac{1}{r}\frac{\partial w^+}{\partial\phi}\right)\right]\sin\phi\right\}_{r=r_p}dz\,d\phi. \qquad [4.65]$$

When the contribution from the inner solution is written in stretched variables and expanded in ε it eventually yields

$$L_t^{(i)} = -2\varepsilon^{-3/2}\int_{-\pi}^{\pi}\int_0^{\zeta/\varepsilon^{1/2}}\left\{\frac{\partial W_0}{\partial X}\cos\phi+\varepsilon\left[Z\frac{\partial V_0}{\partial X}\sin\phi+\left(\frac{\partial W_1}{\partial X}-2Z\frac{\partial U_0}{\partial X}-\frac{\partial U_0}{\partial Z}\right.\right.\right.$$
$$\left.\left.\left.-2Z^2\frac{\partial W_0}{\partial X}-2Z\frac{\partial W_0}{\partial Z}+\frac{1}{8}Z^4\frac{\partial^2 W_0}{\partial X^2}\right)\cos\phi\right]+O(\varepsilon^2)\right\}_{X=H}dZ\,d\Phi. \qquad [4.66]$$

Only terms which are singular in ε in the above expansion have been evaluated. No singular terms are contributed by the outer solution, as follows from [4.65] in conjunction with the nonsingular character of the outer region velocity components. Summing the inner and outer contributions, $L[\varepsilon] = L^{(o)}[\varepsilon;\zeta] + L^{(i)}[\varepsilon;\zeta]$, then leads to,

$$L_t = \tfrac{3}{2}\pi^2\sqrt{2}\,e\eta_0\varepsilon^{-3/2}[1+\varepsilon(\tfrac{1}{60}+\eta_1)] + l_t + O(\varepsilon^{1/2}). \qquad [4.67]$$

The functions $k_t = k_t[e]$ and $l_t = l_t[e]$ have been inserted into expansions [4.64] and [4.67], respectively, to represent the terms which are of order unity with respect to the expansion parameter ε.

For a concentrically-positioned sphere ($e = 0$), the resistance coefficient expansions [4.56], [4.64] and [4.67] simplify to

$$K_t = \frac{9}{4}\pi^2\sqrt{2}\,\varepsilon^{-5/2}\left[1+\frac{77}{60}\varepsilon+\frac{79{,}813}{50{,}400}\varepsilon^2\right]+k_t[0]+O(\varepsilon^{1/2}), \qquad [4.68a]$$

$$L_t = 0, \qquad [4.68b]$$

$$M_t = \frac{9}{4}\pi^2\sqrt{2}\,\varepsilon^{-5/2}\left[1+\frac{157}{60}\varepsilon+\frac{101{,}093}{50{,}400}\varepsilon^2\right]+m_t[0]+O(\varepsilon^{1/2}). \qquad [4.68c]$$

Expansion [4.68a] complements the calculation of the force resistance coefficient performed by Haberman & Sayre (1958) for a single, concentric sphere. Their solution, which is equivalent to the outer solution for axisymmetric fluid motion, produced convergent results for K_t in the range $0 \leq a/R_o \leq 0.8$. In turn, their analysis was extended by Wang & Skalak (1969) to include effects of particle–particle interactions on both the pressure drop and force coefficients for a train of identical, equally-spaced, concentric spheres over the same a/R_o range. Values for the concentric resistance coefficients can be computed to well within an accuracy of 1 per cent over the entire range, $0 \leq a/R_o < 1$, by utilizing the following composite expressions (Bungay 1970), obtained by combining the perturbation expansions [4.68] with asymptotic expansions for $a/R_o \ll 1$ derived from Haberman & Sayre (1958):

$$K_t = \frac{9}{4}\pi^2\sqrt{2}(1-\lambda)^{-5/2}\left[1 - \frac{73}{60}(1-\lambda) + \frac{77{,}293}{50{,}400}(1-\lambda)^2\right] - 22.5083 - 5.6117\lambda - 0.3363\lambda^2 - 1.216\lambda^3 + 1.647\lambda^4,$$

and

$$M_t = \frac{9}{4}\pi^2\sqrt{2}(1-\lambda)^{-5/2}\left[1 + \frac{7}{60}(1-\lambda) - \frac{2227}{50{,}400}(1-\lambda)^2\right] + 4.0180 - 3.9788\lambda - 1.9215\lambda^2 + 4.392\lambda^3 + 5.006\lambda^4.$$

The parameter appearing in the above expressions is the radius ratio, $\lambda = a/R_o$.

Inertial effects

Having obtained the zero-, first- and second-order contributions to the resistance coefficients, we are now in a position to argue that these (singular) coefficients represent the asymptotic solution of the *complete* Navier–Stokes equations, to the order in ε indicated. More precisely, we will demonstrate that the inertial terms in the Navier–Stokes equations affect only the third- and higher-order inner solutions. The argument hinges on nondimensionalizing the Navier–Stokes equations in such a manner that the appropriate Reynolds number appearing therein is independent of the small expansion parameter ε. Explicit arguments will be presented for the translating sphere case. Subsequently, it will be demonstrated that the same general reasoning applies equally well to the rotating and stationary sphere cases too.

When the fluid is stationary ($V_m = 0$), translational motion of the sphere comes about through the action of an external gravitational force applied parallel to the tube axis. Equal and opposite to this is the hydrodynamic force F exerted by the fluid on the sphere. For a fixed value of this force, the translational velocity of the sphere is strongly dependent upon the gap width. Indeed, [3.6] and [4.64] show that as $\varepsilon \to 0$, the translational velocity of the sphere is asymptotically of the form $U_o \sim C\varepsilon^{5/2}|F|/\mu a$, where $C = O(1)$. This suggests the choice of $v_* = |F|/\mu a$ as a characteristic velocity which is independent of ε. For a fixed value of the force, this is the velocity which characterizes the outer flow region.

Accordingly, the outer dependent variables are nondimensionalized via the scheme $\mathbf{v} = v_*\mathbf{v}'$ and $p = (\mu v_*/a)p'$, rather than [4.1] and [4.2]. Concomitantly, the appropriate

Reynolds number is

$$\mathrm{Re}_F = \rho|F|\mu^{-2}, \tag{4.69}$$

which is independent of ε. In place of [4.3], the dynamical equations governing the fluid motion in the outer region when inertial effects are sensible are, therefore,

$$\nabla'^2\mathbf{v}' - \nabla' p' = \mathrm{Re}_F\mathbf{v}'\cdot\nabla'\mathbf{v}'. \tag{4.70}$$

Maintaining the external force constant dictates that the pressure field remain unstretched in the inner flow region, whence $p' = P$. Stretching of the inner velocity components is therefore to be performed according to the relations

$$u' = \varepsilon^2 U, \quad v' = \varepsilon^2 V, \quad w' = \varepsilon^{3/2} W,$$

in present circumstances. These definitions now appear in place of [4.19]–[4.22]. Correspondingly, the requisite expansion of the function of integration appearing in the continuity relations is now,

$$q' = \varepsilon^{5/2}(Q_o + \varepsilon Q_1 + \varepsilon^2 Q_2 + \cdots).$$

Symbolically, then, the Navier–Stokes equations pertaining to flow within the inner region become

$$\nabla^2\mathbf{V} - \nabla P = \varepsilon^3\mathrm{Re}_F\mathbf{V}\cdot\nabla\mathbf{V}. \tag{4.71}$$

For a fixed value of the external force, the inertial terms appearing on the right-hand side of the above are of $O(\varepsilon^3)$ with respect to the viscous and pressure terms. For a fixed value of Re_F, however large, one may expand the inner field $(\mathbf{V}, P)$ as an asymptotic expansion in the small parameter ε. Clearly, the inner fields of orders zero, one, and two, that we have previously calculated on the basis of Stokes' equations ($Re_F = 0$), are solutions of the complete Navier–Stokes equation [4.71]. The singular terms in the resistance-coefficient expansions [4.56], [4.64] and [4.67] are thereby shown to be uninfluenced by inertial effects. Stated alternatively, fluid inertia contributes only terms to the perturbation expansion of the resistance coefficients that are regular in ε. Such terms are small compared with the singular terms in the limit where $\varepsilon \ll 1$.

Inertial effects can, of course, manifest themselves in other ways than through the three resistance coefficients. In particular, they may give rise to a lateral force, in addition to the axially-directed force component of [3.4]. According to the argument given, such a lateral force would necessarily be nonsingular in ε. Nevertheless, the nonsingular contributions could significantly affect the system behavior. If no external force were imposed to counter this lift force, the sphere would undergo lateral migration. No analysis has been performed for a closely-fitting sphere comparable to the lateral migration theory of Cox & Brenner (1968), for a sphere whose radius is small compared with that of the tube.

These remarks regarding inertia pertain to the case of a translating sphere. Similar conclusions, however, apply also to both a rotating and stationary sphere. For the rotational case the corresponding characteristic velocity is $v_* = |T_o|/\mu a^2$, the Reynolds number being $|T_o|\rho/\mu^2 a$.

5. SPHERE ROTATING IN QUIESCENT FLUID

In this section we present asymptotic expressions for the resistance coefficients for a sphere, rotating in accordance with the relation $\mathbf{\Omega} = \mathbf{j}\Omega$, and subject to the conditions that $U_o = V_m = 0$. Derivation of these results follows closely that of the preceding translational case. Equations [4.3] and [4.4] constitute the governing differential equations, wherein the dimensionless local fields are defined as

$$\mathbf{v}^+ = \Omega a \mathbf{v}', \tag{5.1}$$

$$p^+ = \mu\Omega p'. \tag{5.2}$$

In place of [4.5]–[4.9], the boundary and continuity conditions to be employed are now

$$\mathbf{v}' = \mathbf{0} \text{ on } S_p, \quad \mathbf{v}' \to \mathbf{0} \text{ as } |z'| \to \infty, \tag{5.3a, b}$$

$$\mathbf{v}' = \mathbf{i}_r z' \cos\phi - \mathbf{i}_\phi z' \sin\phi - \mathbf{i}_z r' \cos\phi \text{ on } S_w, \quad |z'| \le 1, \tag{5.4}$$

$$\int_{r'_p}^{r'_w} w' r' \,\mathrm{d}r' + \frac{\partial}{\partial\phi}\int\int_{r'_p}^{r'_w} v' \,\mathrm{d}r' \,\mathrm{d}z' = q'[\phi], \quad |z'| \le 1, \tag{5.5}$$

and

$$\int_{-\pi}^{\pi} q' \,\mathrm{d}\phi = 0, \tag{5.6}$$

with $q' = q/\Omega a^3$. By following the singular perturbation procedure previously outlined, this leads eventually to the relations (Bungay 1970),

$$K_r = \frac{3}{2}\pi^2\sqrt{2}\, e\eta_0\varepsilon^{-3/2}\left[1 + \varepsilon\left(\frac{1}{60} + \eta_1\right) + \varepsilon^2\left(-\frac{23}{45} - \frac{7}{15}e^2 - \frac{4}{3}\eta_1 + \frac{16\eta_2}{9\eta_0} - \frac{4\eta_3}{3\eta_0} + \eta_r\right)\right] + k_r + O(\varepsilon), \tag{5.7}$$

$$L_r = \pi^2\sqrt{2}\,\varepsilon^{-1/2}(\eta_0 e^2 + 2\eta_3) + l_r + O(\varepsilon^{1/2}), \tag{5.8}$$

and

$$M_r = \tfrac{3}{2}\pi^2\sqrt{2}\, e\eta_0\varepsilon^{-3/2}[1 + \varepsilon(\tfrac{27}{20} + \eta_1) + \varepsilon^2\eta_r] + m_r + O(\varepsilon^{1/2}), \tag{5.9}$$

in which

$$\eta_3[e] = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos^2\phi}{\tau}\,\mathrm{d}\phi. \tag{5.10}$$

As in the translational case, the unspecified bounded functions $k_r[e]$, $l_r[e]$, $m_r[e]$ and $\eta_r[e]$ have been introduced to represent factors which have not yet been explicitly determined. In the axisymmetric case, the functions k_r and m_r vanish. The function η_3 is reformulated in terms of complete elliptic integrals in the Appendix.

The first of the reciprocity relationships, [3.7], requires that the corresponding terms in expansions [4.67] and [5.7] be identical, implying, for example, that $k_r = l_t$. Both the force and pressure-drop resistance coefficients are identically zero for a concentric sphere. The corresponding torque coefficient becomes

$$L_r = 2\pi^2\sqrt{2}\,\varepsilon^{-1/2} + l_r[0] + O(\varepsilon^{1/2}). \tag{5.11}$$

6. FLOW PAST A STATIONARY SPHERE

When the sphere is motionless relative to the tube wall ($U_o = \Omega = 0$), the behavior of the fluid flowing in the gap between the particle and the wall is described by [4.3] and [4.4], along with the following boundary and continuity conditions:

$$\mathbf{v}' = -2\mathbf{i}_z[1 - (b/R_o)^2 - 2(br/R_o^2)\cos\phi - (r/R_o)^2] \text{ on } S_p, \qquad [6.1]$$

$$\mathbf{v}' = \mathbf{0} \text{ on } S_w, \quad \mathbf{v}' \to \mathbf{0} \text{ as } |z'| \to \infty, \qquad [6.2a, b]$$

$$\int_{r_p'}^{r_w'} wr'\,\mathrm{d}r' + \frac{\partial}{\partial\phi}\int\int_{r_p'}^{r_w'} v'\,\mathrm{d}r'\,\mathrm{d}z' = q' + \{[1 - (b/R_o)^2](r_p/R_o)^2 - \tfrac{4}{3}(b/R_o)(r_p/R_o)^3\cos\phi - \tfrac{1}{2}(r_p/R_o)^4\}, \qquad [6.3]$$

and

$$\int_{-\pi}^{\pi} q'\,\mathrm{d}\phi = 0, \qquad [6.4]$$

in which the dimensionless quantities, denoted by the primes, are defined as

$$\mathbf{v}^+ = V_m\mathbf{v}', \qquad [6.5]$$

$$p^+ = (\mu V_m/a)p', \qquad [6.6]$$

and

$$q = (V_m a^2)q'. \qquad [6.7]$$

Utilization of the perturbation scheme leads to the following asymptotic expressions for the resistance coefficients (Bungay 1970):

$$K_s = \frac{9}{4}\pi^2\sqrt{2}\,\eta_0\varepsilon^{-5/2}\left[1 + \varepsilon\left(\frac{157}{60} + \eta_1\right) + \varepsilon^2\left(\frac{101{,}093}{50{,}400} + \eta_t\right)\right] + k_s + O(\varepsilon^{1/2}), \qquad [6.8]$$

$$L_s = \tfrac{3}{2}\pi^2\sqrt{2}\,e\eta_0\varepsilon^{-3/2}[1 + \varepsilon(\tfrac{27}{20} + \eta_1) + \varepsilon^2\eta_r] + l_s + O(\varepsilon), \qquad [6.9]$$

$$M_s = \frac{9}{4}\pi^2\sqrt{2}\eta_0\varepsilon^{-5/2}\left[1 + \varepsilon\left(\frac{79}{20} + \eta_1\right) + \varepsilon^2\left(\frac{301{,}573}{50{,}400} + \frac{7}{15}e^2 + \frac{4}{3}\eta_1 + \eta_t\right)\right] + m_s + O(\varepsilon^{1/2}). \qquad [6.10]$$

The leading terms of [6.8] and [6.9] are in agreement with those of [4.56] and [5.9], as required by the reciprocity relationships [3.8] and [3.9]. Likewise, the latter formulae require that the unknown terms of order unity satisfy the interrelationships $k_s = m_t$ and $l_s = m_r$.

Setting $e = 0$ in [6.8]–[6.10] produces the following values of the resistance coefficients for a stationary, concentric sphere:

$$K_s = \frac{9}{4}\pi^2\sqrt{2}\,\varepsilon^{-5/2}\left[1 + \frac{157}{60}\varepsilon + \frac{101{,}093}{50{,}400}\varepsilon^2\right] + k_s[0] + O(\varepsilon^{1/2}), \qquad [6.11a]$$

$$L_s = 0, \qquad [6.11b]$$

$$M_s = \frac{9}{4}\pi^2\sqrt{2}\,\varepsilon^{-5/2}\left[1 + \frac{79}{20}\varepsilon + \frac{301{,}573}{50{,}400}\varepsilon^2\right] + m_s[0] + O(\varepsilon^{1/2}). \qquad [6.11c]$$

As in the translational case, values of the force coefficient were computed by Haberman & Sayre (1958) for a single, concentric sphere, and by Wang & Skalak (1969) for a sphere in an infinite train of concentric spheres. Convergence difficulties restricted the validity of both sets of calculations to the range $0 \leq a/R_o \leq 0.8$. Wang & Skalak (1969) also determined values for the pressure-drop coefficient in this range. Both coefficients can be evaluated to within an accuracy of 1 per cent for any radius ratio, $0 \leq a/R_o < 1$, from the following *ad hoc* composite expressions (Bungay 1970), obtained by combining the perturbation expansions [6.11] with asymptotic expansions for $a/R_o \ll 1$ derived from Haberman & Sayre (1958):

$$K_s = \frac{9}{4}\pi^2\sqrt{2}(1-\lambda)^{-5/2}\left[1 + \frac{7}{60}(1-\lambda) - \frac{2227}{50{,}400}(1-\lambda)^2\right] + 4.0180 - 3.9788\lambda - 1.9215\lambda^2 + 4.392\lambda^3 + 5.006\lambda^4,$$

and $$M_s = \frac{9}{4}\pi^2\sqrt{2}(1-\lambda)^{-5/2}\left[1 + \frac{29}{20}(1-\lambda) + \frac{97{,}453}{50{,}400}(1-\lambda)^2\right] - 62.2680 - 18.509\lambda - 12.156\lambda^2 + 9.15\lambda^3 + 43.17\lambda^4,$$

in which $\lambda = a/R_o$.

7. HOMOGENEOUS NEUTRALLY-BUOYANT SPHERE IN A POISEUILLE FLOW

The hydrodynamic force and torque vanish for a homogeneous, neutrally-buoyant sphere. In these circumstances, [3.6] yields the following particle/fluid velocity ratios:

$$\frac{U_o}{V_m} = \frac{K_sL_r - K_rL_s}{K_tL_r - K_rL_t}, \tag{7.1}$$

$$\frac{\Omega a}{V_m} = \frac{K_tL_s - K_sL_t}{K_tL_r - K_rL_t}, \tag{7.2}$$

in addition to the pressure-drop force expression,

$$\Delta p^+ A = \mu V_m a\left(M_s - M_t\frac{U_o}{V_m} - M_r\frac{\Omega a}{V_m}\right). \tag{7.3}$$

Substitution of the various resistance-coefficient perturbation expansions from Sections 4 to 6 into [7.1]–[7.3] yields

$$\frac{U_o}{V_m} = 1\bigg/\left[1 - \frac{4}{3}\varepsilon + \frac{2}{3}\varepsilon^{3/2}e\Lambda_t + \varepsilon^2\left(\frac{58}{45} + \frac{1}{5}e^2 + \frac{16\eta_2}{9\eta_0}\right) + O(\varepsilon^{5/2})\right], \tag{7.4}$$

$$\frac{\Omega a}{V_m} = \frac{\varepsilon^{1/2}\Lambda_t + \varepsilon e + O(\varepsilon^{3/2})}{1 + \varepsilon^{1/2}\Lambda_r + O(\varepsilon)}, \tag{7.5}$$

and $$\Delta p^+ R_o/\mu V_m = 4\pi\sqrt{2}\,\eta_2\varepsilon^{-1/2} + O(1), \tag{7.6}$$

in which $\Lambda_t[e] = (l_s - l_t)/2\pi^2\sqrt{2}\eta_3$, and $\Lambda_r[e] = l_r/2\pi^2\sqrt{2}\,\eta_3$. For the concentric position $(e = 0)$, the coefficient Λ_t is zero. There is evidence (Bungay 1970) in the evaluation of the torque coefficients L_s and L_t to suggest that the functions l_s and l_t are identical. Hence Λ_t may be identically zero for all e, in which case the velocity ratios would be

$$\frac{U_o}{V_m} = 1\bigg/\left[1 - \frac{4}{3}\varepsilon + \varepsilon^2\left(\frac{58}{45} + \frac{1}{5}e^2 + \frac{16\eta_2}{9\eta_0}\right) + O(\varepsilon^{5/2})\right], \qquad [7.7]$$

$$\frac{\Omega a}{V_m} = \varepsilon e + O(\varepsilon^{3/2}). \qquad [7.8]$$

In the limit $e \to 1$ the η_1 functions may be replaced by asymptotic expressions [A.9]–[A.12] of the Appendix. For ε held constant the resistance coefficients then adopt the limiting forms,

$$K_t \sim 4\pi\varepsilon^{-1/2} \ln\left(\frac{32}{1-e}\right), \qquad [7.9]$$

$$L_r \sim 4\pi\varepsilon^{-1/2} \ln\left(\frac{32}{1-e}\right), \qquad [7.10]$$

$$K_r = L_t \sim -\frac{4}{3}\pi\varepsilon^{+1/2} \ln\left(\frac{32}{1-e}\right), \qquad [7.11]$$

$$K_s = M_t \sim \tfrac{135}{256}\pi^3\varepsilon^{-5/2}, \qquad [7.12]$$

$$L_s = M_r \sim \tfrac{45}{128}\pi^3\varepsilon^{-3/2}, \qquad [7.13]$$

$$M_s \sim \tfrac{135}{256}\pi^3\varepsilon^{-5/2}, \qquad [7.14]$$

for $e \to 1$. The above expressions indicate that [7.1]–[7.3] can be approximated by

$$U_o/V_m \sim 135\pi^2/1024\varepsilon^2 \ln\left(\frac{32}{1-e}\right), \qquad [7.15]$$

$$\Omega a/V_m \sim 135\pi^2/1024\varepsilon \ln\left(\frac{32}{1-e}\right), \qquad [7.16]$$

$$\Delta p^+ R_o/\mu V_m \sim 135/256\pi^2\varepsilon^{5/2}, \qquad [7.17]$$

in this limit.

It appears from [7.7] that the translational velocity is quite insensitive to changes in the lateral position of the sphere for most $e < 1$. When the sphere moves along the tube axis its velocity is

$$\frac{U_o}{V_m} = 1\bigg/\left[1 - \frac{4}{3}\varepsilon + \frac{46}{15}\varepsilon^2 + O(\varepsilon^{5/2})\right]. \qquad [7.18]$$

Displacement of the sphere to eccentric positions decreases its velocity only slightly, unless e is very close to unity. Even when $e = 0.98$ the velocity ratio in [7.7] decreases by only 4 per cent from the concentric value given by [7.18] for $\varepsilon = 0.1$. Thus, for most eccentric positions, the sphere translates faster than the mean fluid velocity. On the other hand, [7.15] suggests that the sphere velocity would be much less than the fluid velocity for a sphere which is nearly in contact with the tube wall. Theoretically, if such contact did exist, the sphere would remain stationary, since the velocity ratios in [7.15] and [7.16] reduce to zero for $e = 1$.

In the concentric position ($e = 0$) a neutrally-buoyant sphere translates without rotation. According to [7.15] and [7.16] a sphere in the fully eccentric position ($e = 1$) neither translates nor rotates. Hence, the perturbation analysis indicates that the magnitude of the angular velocity attains a maximum value at an intermediate lateral position. The direction of rotation always corresponds to that for rolling along the nearer side of the tube. For any lateral position the circumferential linear velocity at the sphere surface due to rotation is always small compared with that due to translation. From [7.15] and [7.16] one obtains $\Omega a/U_o \sim \varepsilon$ for $e \to 1$. This limiting behavior would require a relatively high degree of slip, since $\Omega a/U_o = 1$ for pure rolling motion.

The minimum pressure drop is occasioned by a concentrically positioned sphere, namely,

$$\Delta p^+ R_o/\mu V_m = 4\pi\sqrt{2}\,\varepsilon^{-1/2} + O(1), \qquad [7.19]$$

for $e = 0$. This result was first derived by Hochmuth & Sutera (1970) using lubrication-theory methods. These authors estimated the constant term of $O(1)$ in [7.19] to be -31.5 when the expansion parameter was chosen to be $(R_o - a)/R_o$ rather than ε. Lubrication theory simulates the inner solution of the perturbation analysis. In general, only the leading term of such an asymptotic expansion can be rigorously calculated by the former method. Rigorous determination of the constant term in [7.19] necessitates knowledge of at least the leading terms of the outer expansion. Their calculation transcends standard lubrication theory. Accordingly, the value obtained for the constant term by Hochmuth & Sutera (1970) should be regarded as empirical. It does appear, however, to be of the correct order-of-magnitude, based upon their comparison with available experimental evidence.

Lateral displacement of the sphere from the concentric position leads to an increase in additional pressure drop, all other things being equal. This increase could be very considerable. The perturbation analysis predicts a maximum neutrally-buoyant pressure drop given by [7.17] for a sphere in contact with the wall. If this pressure drop were realized it would be larger by a factor of $135\sqrt{2\pi}/2048\varepsilon^2 = 0.2929\varepsilon^{-2}$ than the minimum value at the same flow rate.

8. SEDIMENTATION OF A SPHERE IN A VERTICAL TUBE

If the sphere density ρ_p is not matched to the density ρ of the surrounding fluid, the sphere experiences a net external force, $\frac{4}{3}\pi a^3(\rho_p - \rho)\mathbf{g}$, equal and opposite to the hydrodynamic force, $\mathbf{F}$, with $\mathbf{g}$ the acceleration of gravity vector. Consider the effect of this external force when the tube is oriented vertically as in figure 1, so that $\mathbf{g} = -g\mathbf{k}$. Let the tube be sealed at its bottom ($V_m = 0$), and let the sphere be homogeneous and freely rotating, such that it suffers no external or hydrodynamic torques. The circumstances thus described are commonly realized in the falling-ball viscometer.

Theoretical behavior

Particle velocities obtained from the solution of the matrix equation [3.6] for this case are

$$U_o = -\frac{F}{\mu a}\left(\frac{L_r}{K_t L_r - K_r L_t}\right), \quad [8.1]$$

$$\Omega = \frac{F}{\mu a^2}\left(\frac{K_r}{K_t L_r - K_r L_t}\right). \quad [8.2]$$

Concomitantly, the pressure drop is

$$\frac{\Delta p A}{F} = \frac{K_s L_r - K_r L_s}{K_t L_r - K_r L_t}. \quad [8.3]$$

To evaluate the particle velocities and pressure drop for any lateral position $0 \le e \le 1$, one can calculate the resistance coefficients from the expansions of Sections 4–6 and substitute the resulting values into [8.1]–[8.3]. Alternatively, for $e < 1$ one can employ the expressions,

$$U_o = -\frac{2\sqrt{2}\,F}{9\pi^2 \mu a}\left(\frac{1}{\eta_0} + \frac{e^2}{2\eta_3}\right)\varepsilon^{5/2} + O(\varepsilon^3), \quad [8.4]$$

$$\Omega = \frac{\sqrt{2}\,eF\varepsilon^{3/2}}{6\pi^2 \mu a^2 \eta_3} + O(\varepsilon^2), \quad [8.5]$$

$$\frac{\Delta p A}{F} = 1\Big/\left[1 - \frac{4}{3}\varepsilon + \varepsilon^2\left(\frac{58}{45} + \frac{1}{5}e^2 + \frac{16\eta_2}{9\eta_0}\right) + O(\varepsilon^{5/2})\right]. \quad [8.6]$$

In the limit $e \to 1$, use of [7.9]–[7.13] in [8.1]–[8.3] leads to

$$U_o \sim -F\varepsilon^{1/2}\Big/4\pi\mu a \ln\left(\frac{32}{1-e}\right), \quad [8.7]$$

$$\Omega \sim -F\varepsilon^{3/2}\Big/12\pi\mu a^2 \ln\left(\frac{32}{1-e}\right), \quad [8.8]$$

$$\frac{\Delta p A}{F} \sim 135\pi^2\Big/1024\varepsilon^2 \ln\left(\frac{32}{1-e}\right). \quad [8.9]$$

A concentric sphere ($e = 0$) settles without rotation. If displaced to an intermediate position, the sphere descends more rapidly. Simultaneously, it undergoes rotation in a direction opposite to that for rolling along the nearer side of the tube wall. According to [8.4], the settling velocity increases monotonically with lateral position until about $e = 0.98$, in which position the sphere settles approximately 2.1 times faster than for $e = 0$. For $e > 0.98$, the rate of settling decreases, as does the angular velocity. In the limit where $e = 1$, a sphere in contact with the wall is theoretically prohibited from moving ($U_o = \Omega = 0$), since the finite external gravity force is incapable of generating the requisite infinite shear rate, and concomitant infinite stresses at the contact point.

The perturbation solution further predicts that at some lateral position e, close to unity, the direction of rotation changes, as indicated by the difference in algebraic signs between the right-hand sides of [8.5] and [8.8]. However, the motion would be very different from rolling along the wall. From [8.7] and [8.8] one finds $\Omega a/U_o = (1/3)\varepsilon \ll 1$ in the limit $e \to 1$, whereas for rolling without slip, $\Omega a/U_o = 1$. This limiting behavior is in qualitative agreement with the theoretical prediction of Goldman, Cox & Brenner (1967) for a sphere settling parallel to a nearby plane boundary.

The leading terms of [8.4] and [8.5], of orders $\varepsilon^{5/2}$ and $\varepsilon^{3/2}$, respectively, have previously been derived by Christopherson & Dowson (1959), and by Floberg (1968) from lubrication-theory arguments. The lead term in [8.4] primarily reflects the difference in pressure between the upstream and downstream surfaces of the sphere. These lubrication-theory treatments are deficient in that they fail to demonstrate the singular effect of the local shear stresses near the point of closest approach to the wall, manifested by [8.7] and [8.8].

The ratio of pressure drop force to hydrodynamic force, as given by [8.3], is identical to the ratio of translational to mean velocities for a neutrally-buoyant, homogeneous sphere (cf. [7.1]). The existence of this equality was demonstrated generally by Bungay & Brenner (1973). Although the settling velocity undergoes a two-fold change with lateral position in the range $0 \leq e \leq 0.98$, according to [8.6] the pressure drop remains essentially constant over this same range. If the sphere becomes motionless in the limit $e \to 1$, as required by [8.7] and [8.8], then the pressure drop [8.9] vanishes.

Comparison with experimental observations

As discussed in Section 3, no tendency exists for a sphere settling in a vertical tube to move laterally in the absence of inertial effects. Several investigators have, however, noted a decided tendency for sedimenting spheres to migrate away from the tube axis during the course of experiments for which $\varepsilon \ll 1$.

For particle Reynolds numbers $Re_p = 2aU_o\rho/\mu$ in the range $10^{-3} < Re_p < 10^{-1}$, Christopherson & Dowson (1959) observed that spheres invariably moved toward the tube wall, rotating as they settled. For a particular ball and tube pair, characterized by $\varepsilon = 0.008$, the settling velocity was found to be nearly constant over a decade change in Reynolds number. Furthermore, this velocity was very close to the maximum predicted by the leading term of [8.4]. In this same Re_p range, larger clearances $(0.0088 < \varepsilon < 0.081)$ yielded settling velocities intermediate between the leading-term maximum and concentric minimum. Both the direction and the order-of-magnitude of the rotary velocity agreed with the leading-term prediction of [8.5]. Visually, the eccentricities appeared to be near unity. On this basis, Christopherson & Dowson (1959) concluded that the spheres ultimately adopt a particular eccentric position, corresponding to a maximum settling velocity. They argued that the eccentric location corresponding to this maximum value varied due to the increased influence of the higher-order terms at the larger clearances.

Earlier experimental work by McNown *et al.* (1948) corroborates the tendency for closely-fitting spheres to migrate to eccentric positions for $Re_p > 10^{-3}$. In contrast to Christopherson & Dowson (1959), these authors found that the spheres settled concentrically at $Re_p < 10^{-3}$. Though eccentric positions were observed to result in sphere rotation, McNown (1951) implies the direction of rotation to be opposite to that recorded by Christopherson & Dowson (1959).

Both groups of investigators took precautions to insure vertical alignment of their tubes. In "rolling-ball" viscometers the tube is intentionally inclined at some angle from the vertical. The component of the gravitational force perpendicular to the vessel axis presumably maintains the eccentricity nominally at $e = 1$. A number of studies have been reported pertaining to the operation of these tilted viscometers. Depending upon the angle of inclination, and possibly other factors which have yet to be delineated, the balls were observed to adopt various modes of rotational motion. Zolotykh (1962) implies that in viscometers possessing a typical tilt of 10° from the vertical, balls in the range $\varepsilon < 0.20$ will roll without slipping as they travel down the tube; that is, the angular/translational speed ratio for such balls is $\Omega a/U_o = 1$. This behavior is corroborated by the findings of McNown (1956). Block (1940), on the other hand, ran repeated tests at various angles of inclination. When the tube was tilted far from the vertical, he likewise found rolling without slipping. Under the same conditions, Block noted that as the tube approached a vertical position, slipping would occur, such that $\Omega a/U_o < 1$. Block did not state that under certain conditions the rotation would cease or reverse direction. However, Floberg (1968) found, for clearances of $0.0027 < \varepsilon < 0.012$, that at small inclinations of 3–4° from the vertical, the balls appeared to settle without rotating. On the other hand, at larger clearances ($0.024 < \varepsilon < 0.094$) the settling balls rotated in a sense opposite to that for rolling, in some cases continuing to rotate at angle of inclination as large as 11.5°.

Thus, there exists a seeming disparity in the experimental observations for the angular velocity of a settling sphere nominally in the fully eccentric position. One must refer to the eccentricity as nominal in describing the experimental operations, since none of the investigators cited make reference to direct measurements of eccentric position. It seems reasonable to suppose that some finite separation exists between the sphere surface and tube wall when slipping occurs, and that the separation distance, however small, may crucially determine the character of the rotation. As noted earlier, the perturbation solution does suggest a change in the direction of rotation at an eccentricity very close to, but not equal to, unity. According to [8.7] and [8.8], for $e \approx 1$ rotational movement may be difficult to detect in comparison to the translational motion for closely-fitting spheres.

All experiments are in substantial agreement that the translational settling velocity in the inclined viscometer is of the order of the leading term of [8.4]. This contrasts with the likely theoretical prediction of zero velocity at $e = 1$, lending credence to the view that the actual eccentricity is somewhat less than unity. The lead term reflects the pressure difference prevailing in the fluid above and below the sphere, which would be but little affected by small discrepancies in eccentricity.

These theoretical predictions could well be vitiated by surface asperities, frictional contact, or other nonhydrodynamic factors. In the related problem of a sphere moving parallel to a nearby plane wall bounding a semi-infinite liquid, Goldman, Cox & Brenner (1967) considered cavitation to be the most likely explanation for the observed inconsistency between their theoretical analysis and experimental measurements. Pressure peaking in the region of minimum separation is common to both problems. Relative to the pressure at the midplane, $z = 0$, the theory predicts a sharp positive peak in pressure on one side of the midline, and a symmetrical negative peak in pressure on the other. Hence, a condition exists which is conducive to cavitation, at least in the case of liquids. Floberg (1968) found visual evidence of flow patterns in the region of minimum separation which he ascribed to cavitation. Possible consequences of this phenomenon are mitigation of the singular behavior of the force or torque as $e \to 1$, and creation of a radially directed "lift" force.

Final remarks concern a facet of the asymptotic solution that may not have been recognized in some of the earlier studies. Although the form of the leading term in [8.4] was evaluated by several investigators using lubrication-theory arguments, no notice was taken of the fact that such approximation techniques cannot discriminate between the alternative clearance parameters, ε and $\delta = (R_o - a)/R_o$. Christopherson & Dowson (1959) as well as Floberg (1968), employ ε in their analysis, whereas McNown *et al.* (1948, 1951, 1956) use δ. Zolotykh (1962) arrived at an empirical equation in δ from dimensional analysis arguments. Since the leading term is proportional to the 5/2 power of the clearance parameter, the choice of ε over δ can make an appreciable difference in the predicted value. For example, for $\varepsilon = 0.05$ (which is in the range common to the experiments just cited), the value of the lead term is approximately 14 per cent higher based on ε than on δ. The complete expansion should converge to a result which is independent of the particular clearance parameter employed. Thus, the 14 per cent disparity would indicate that the higher order terms are significant. The utility of the perturbation technique consists of providing a rational procedure for generating the correct higher-order terms in the asymptotic expansion. Further progress with the present problem requires a more detailed analysis of the outer expansion.

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APPENDIX

Evaluation of the η_i functions

Variation of the resistance coefficients with lateral position for a closely-fitting sphere is embodied in the η_i functions defined in [4.50], [4.53], [4.62] and [5.10]. Each of these are even functions of the eccentricity parameter e of [4.10], i.e. $\eta_i[-e] = \eta_i[e]$. Utilization of the substitition $\phi = 2\theta$, in the manner of Christopherson & Dowson (1959), yields the equivalent forms

$$\eta_0[e] = \tfrac{1}{2}\pi(1 - \tfrac{1}{2}m)^{5/2}\left[\int_0^{\pi/2} \sigma^5 \, d\theta\right]^{-1}, \qquad [A.1]$$

$$\eta_1[e] = \frac{193}{60} - \frac{\eta_0}{30\pi(1 - \frac{1}{2}m)^{7/2}}\left[90(1 - m)\int_0^{\pi/2} \sigma^3 \, d\theta + 103\int_0^{\pi/2} \sigma^7 \, d\theta\right], \qquad [A.2]$$

and

$$\eta_2[e] = \frac{2}{\pi}(1 - \tfrac{1}{2}m)^{1/2}\int_0^{\pi/2} \frac{d\theta}{\sigma}, \qquad [A.3]$$

in which $\sigma = (1 - m\sin^2\theta)^{1/2}$ and $m = 2e/(1 + e)$. Parameters e and m are equal at the upper and lower bounds of their respective intervals, $0 \le e \le 1$ and $0 \le m \le 1$. The remaining η_i function was transformed by Christopherson & Dowson (1959) into the following identity, involving the complete elliptic integrals $K[m]$ and $E[m]$ of the first and second kinds, respectively, with parameter m:

$$\eta_3[e] = \frac{4(1 - \frac{1}{2}m)^{1/2}}{3\pi m^2}\{(8 - 8m + 3m^2)K[m] - 8(1 - \tfrac{1}{2}m)E[m]\}. \qquad [A.4]$$

Integration formula 2.582 − 1 of Gradshteyn and Ryzhik (1965) may be employed to establish that

$$\int_0^{\pi/2} \frac{d\theta}{\sigma} = K[m], \qquad [A.5]$$

$$\int_0^{\pi/2} \sigma^3 \, d\theta = \tfrac{4}{3}(1 - \tfrac{1}{2}m)E[m] - \tfrac{1}{3}(1 - m)K[m], \qquad [A.6]$$

$$\int_0^{\pi/2} \sigma^5 \, d\theta = \tfrac{1}{15}\{[32(1 - \tfrac{1}{2}m)^2 - 9(1 - m)]E[m] - 8(1 - \tfrac{1}{2}m)(1 - m)K[m]\}, \qquad [A.7]$$

and

$$\int_0^{\pi/2} \sigma^7 \, d\theta = \tfrac{12}{7}(1 - \tfrac{1}{2}m)\int_0^{\pi/2} \sigma^5 \, d\theta - \tfrac{5}{7}(1 - m)\int_0^{\pi/2} \sigma^3 \, d\theta. \qquad [A.8]$$

Equations [A.1]–[A.8] permit evaluation of all four η_i functions from tabulated values of the complete elliptic integrals. Results are presented graphically in figure 2. For a sphere in contact with the wall the first two functions attain the finite limiting values,

$$\eta_0[1] = \frac{15\pi\sqrt{2}}{128}, \qquad [A.9]$$

$$\eta_1[1] = \tfrac{23}{84}. \qquad [A.10]$$

The remaining two functions are unbounded in the limit $e \to 1$, for which their asymptotic forms are

$$\eta_2[e] \sim \frac{\sqrt{2}}{2\pi}\ln\left(\frac{32}{1 - e}\right), \qquad [A.11]$$

$$\eta_3[e] \sim \frac{\sqrt{2}}{\pi}\ln\left(\frac{32}{1 - e}\right). \qquad [A.12]$$

Sommaire—Des techniques de perturbation unique sont utilisées pour étudier l'écoulement lent, assymétrique autour d'une sphère placée eccentriquement dans un tube long, cylindrique circulaire rempli de fluide visqueux. Les résultats s'appliquent aux situations dans lesquelles la sphère occupe virtuellement la section entière du cylindre, de sorte que l'espace entre la particule et la paroi du tube est petit partout par rapport à la sphère et aux rayons du tube. La technique est une amélioration par rapport aux analyses classiques de "théorie de lubrification".

Des expansions asymtotiques, valables pour de petits espaces sans dimensions, sont obtenues pour la force hydrodynamique, le couple et la perte de pression pour un écoulement au long d'une sphère immobile, ainsi que dans le cas d'une sphère en translation ou en rotation dans un fluide autrement immobile. Ces expansions sont utilisées pour prédire le comportement macroscopique d'une sphère neutralement flottante en suspension dans un écoulement de Poisseuille, et d'une sphère de sédimentation dans un tube vertical.

Les résultats trouvent une application dans l'écoulement capillaire du sang, dans le transport par pipelines de matériaux enrobés et dans les viscosimètres à bille tombante.

Auszug—Singuläre Störungsverfahren werden zur Untersuchung von langsamem, asymmetrischen Fluß um eine Kugel benutzt, die sich exzentrisch innerhalb eines langen, kreisförmigen, zylindrischen, mit viskoser Flüssigkeit gefüllten Rohres befindet. Die Ergrbnisse treffen für Situationen zu, in denen die Kugel praktisch den ganzen Querschnitt des Zylinders einnimmt, sodaß der Spielraum zwischen dem Teilchen und der Rohrwand, verglichen mit den Radien der Kugel uüd des Rohres, überall klein ist. Das Verfahren stellt eine Verbesserung gegenüber den üblichen Analysen der "Abschmiertheorie" dar.

Es werden asymptotische, für kleine dimensionslose Spielräume gültige Expansionen für die hydrodynamische Kraft, Drehmoment und Druckabfall für Fluß, an einer feststehenden Kugel vorüber, erhalten, sowie für den Fall einer sich translatorisch bewegenden oder drehenden Kugel in einem andernfalls ruhigen Feld. Diese Expansionen werden benutzt, um das makroskopische Verhalten sowohl einer neutral schwimmenden, in einem Poiseuille Fluß schwebenden Kugel und einer Sedimentkugel in einem vertikalen Rohr vorauszusagen.

Die Ergebisse werden in Kapillarblutfluß, Rohrleitungsbeförderung von verkapselten Materialien und Fallkugelviskometern verwandt.

Резюме—Для исследования медленного асимметричного течения вокруг шара, расположенного эксцентрически внутри длинной круглой, цилиндрической трубы наполненной вязкой текучей средой, применялся сингулярный метод возмущения. Результаты относятся к положениям в которых шар фактически занимает почти что все сечение цилиндра, так что зазор между частицей и стенкой трубы везде небольшой по сравнению как с радиусом шара так и с радиусом трубы. Этот метод является более совершенным, чем стандартный анализ "теории смазки".

Асимптотические расширения, действительные для малых безразмерных зазоров, получили для гидродинамической силы, для крутящего момента и для падения давления потока проходящего мимо неподвижного шара, также как и для случая где шар перемещается или вращается в спокойной в других отношениях текучей среде. Эти объяснения применяются для предсказания макроскопического поведения как нейтрально плавучего шара, так и шара взвешенного в течении Пуазейля и для осаждающегося в вертикальной трубе.

Результаты можно использовать для капиллярного течения крови, для трубопроводного транспорта заключенных в капсулы материалов и для вискозиметров с падающим шариком.